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Corotational Total Lagrangian Formulation for Three-Dimensional Beam Element

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A corotational total Lagrangian formulation of beam element is presented for the nonlinear analysis of three-dimensional beam structures with large rotations but small strains. The nonlinear coupling among the bending, twisting, and stretching deformations is considered. All of the element deformations and equations are defined in body-attached element coordinates. Three rotation parameters are proposed to determine the orientation of the element cross section. Two sets of element nodal parameters termed explicit nodal parameters and implicit nodal parameters are introduced. The explicit nodal parameters are used in the assembly of the system equations from the element equations and chosen to be three components of the incremental translation vector and three components of the incremental rotation vector. The implicit nodal parameters are used to determine the deformations of the beam element and chosen to be three components of the total displacement vector and nodal values of the three rotation parameters. The element internal nodal forces corresponding to the implicit nodal parameters are obtained from the virtual work principle. Numerical examples are presented and compared with the numerical and experimental results reported in the literature to demonstrate the accuracy and efficiency of the proposed method.

I. Introduction

THREE-DIMENSIONAL beams are very important structural elements in all types of engineering systems. In many applications, these beam elements undergo finite rotations that require a nonlinear formulation to their structural analysis. The development of new and efficient formulations for nonlinear analysis of beam structures has attracted the study of many researchers in recent years. Based on different kinematic assumptions, different alternative formulation strategies and procedures to accommodate large rotation capability during the large deformation process have been presented.[1-20] The kinematic assumptions used in Refs. 4, 5, 7, 11, and 17–19 are based on Timoshenko's hypothesis: the effect of stretching, bending, torsion, and transverse shear are taken into account. However, the warping of cross sections are only considered in Refs. 4, 7, and 19. In Refs. 2 and 13–15, the Euler-Bernoulli hypothesis is employed, and the warping of the cross sections is used to consider coupled bending-torsion phenomena. The formulations, which have been used in the literature, might be divided into three categories: total Lagrangian (TL) formulation,[2-5,7,10,11,13-15,17-20] updated Lagrangian (UL) formulation,[3,5,12] and corotational (CR) formulation.[6,8,9,12] It should be noted that within the corotating system either a TL or a UL formulation, or even a formulation based on a small deflection theory, may be employed. In Refs. 8 and 9, a simple and effective CR formulation of beam element and numerical procedure is proposed for geometrically nonlinear analysis of spatial frames using incremental-iterative methods. This method is proven to be very effective by numerous numerical examples studied in Refs. 8 and 9. However, the nonlinear coupling among the bending, twisting, and stretching deformations, which are essential for the solution of lateral and torsional buckling of beam structures, is not considered in Refs. 8 and 9.

The objective of this paper is to present a corotational total Lagrangian formulation for three-dimensional beam element that includes the nonlinear coupling among the bending, twisting, and stretching deformations.

The numerical algorithm used here is an incremental-iterative method based on the Newton-Raphson method combined with constant arc length of incremental displacement vector.[8,21] To improve the convergence properties of the equilibrium iterations, the two-cycle iteration scheme proposed in Ref. 22 is employed. Numerical examples are presented and compared with the numerical and experimental results reported in the literature to demonstrate the accuracy and efficiency of the proposed method.

II. Nonlinear Formulation

The following assumptions are made in the derivation of the nonlinear behavior.

1) The beam is prismatic and slender, and the Euler-Bernoulli hypothesis is valid.

2) The centroid and the shear center of the cross section coincide.

3) The unit extension and twist rate of the beam element along the deformed axis of the shear center are constants.

4) The cross section of the beam element does not deform in its own plane, and strains within this cross section can be neglected.

5) The out-of-plane warping of the cross section is the product of the twist rate of the beam element and the Saint Venant warping function for a prismatic beam of the same cross section.

6) The deformations of the beam element are small.

Coordinate Systems

In this paper, a corotational total Lagrangian formulation is adopted. To describe the motion of the system, we define three sets of coordinate systems:

1) A fixed global set of coordinates, X_i, $i = 1, 2, 3$ (Fig. 1); the nodal coordinates, incremental nodal displacements and rotations, and equilibrium equations of the system are defined in these coordinates.

2) Element coordinates, x_i, $i = 1, 2, 3$ (Fig. 1); a set of element coordinates is associated with each element. The origin of this coordinate system is located at node 1, the x_1 axis is chosen to pass through the two end nodes of the element, and the x_2 and x_3 axes are determined from the orientations of

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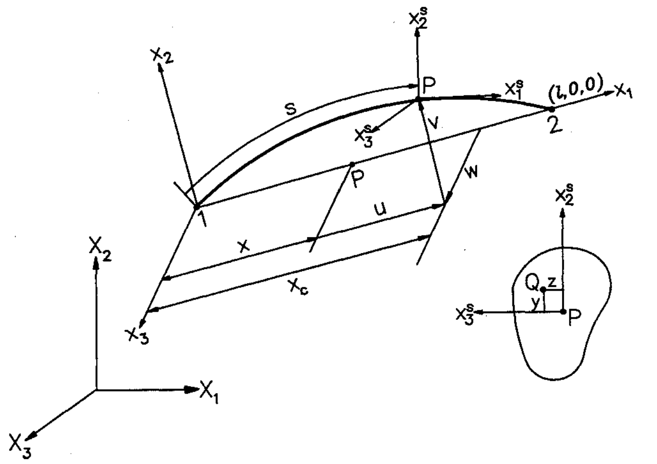

Fig. 1 Coordinate systems.

unwarped end cross sections of the beam element in a manner to be specified later. The deformations, internal nodal forces, and stiffness matrices of the individual elements are defined in terms of these coordinates.

3) Element cross section coordinates, x_i^s, $i = 1, 2, 3$ (Fig. 1); a set of element cross section coordinates is associated with each cross section of the beam element. The origin of this coordinate system is rigidly tied to the shear center of the unwarped cross section. The x_1^s axes are chosen to coincide with the normal of the corresponding unwarped cross section, and the x_2^s and x_3^s axes are chosen to be the principal directions of the unwarped cross section. In this paper the element deformations are determined by rotations of these coordinate systems relative to the element coordinate system.

For convenience of the later discussion, the term "rotation vector" is used to represent a finite rotation. Figure 2 shows a vector $\boldsymbol{b}$ that as a result of the application of a rotation vector $\phi \boldsymbol{a}$ is transported to a new position $\boldsymbol{b}'$. The relation between $\boldsymbol{b}$ and $\boldsymbol{b}'$ may be expressed as[23]

$$\boldsymbol{b}' = \cos\phi \boldsymbol{b} + (1 - \cos\phi)(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{a} + \sin\phi(\boldsymbol{a} \times \boldsymbol{b}) \tag{1}$$

where ϕ is the angle of counterclockwise rotation, and $\boldsymbol{a}$ is the unit vector along the axis of rotation.

In this study, { } denotes column matrix, and $(\ )_{,s} = (\ )'$ denotes differentiation with respect to s.

Let the position vectors of a point P (see Fig. 1) on the centroid axis of the beam member before deformation and after deformation be $\{x, 0, 0\}$ and $\{x_c(s), v(s), w(s)\}$, respectively, in the element coordinate system, where s is the arc length of the deformed centroid axis measured from one end of the beam to this point. With the addition of assumption 3, we may obtain

$$s = (1 + \epsilon_0)x \tag{2}$$

$$\epsilon_0 = (S - L)/L \tag{3}$$

where ϵ_0 is the unit extension of the undeformed arc length of the centroid axis, and S and L are the arc length of the deformed and undeformed centroid axis of the beam element, respectively. In this study $x_c(s)$ is expressed as

$$x_c(s) = u_{11} + \frac{S}{2}\int_{-1} (1 - \theta_2^2 - \theta_3^2)^{1/2}\, \mathrm{d}\xi \tag{4}$$

where

$$\theta_2 = \frac{-\mathrm{d}w(s)}{\mathrm{d}s} \tag{5}$$

$$\theta_3 = \frac{\mathrm{d}v(s)}{\mathrm{d}s} \tag{6}$$

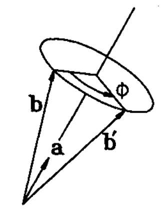

Fig. 2 Finite rotation of vector.

and where u_{11} is the displacement of node 1 in the x_1 direction, and $\xi = -1 + 2s/S$ is a nondimensional coordinate. Due to the definition of the element coordinates, $u_{11} = 0$. Let ℓ denote the chord length of the deformed centroid axis. Making use of Eq. (4), we obtain

$$S = 2\ell \Big/ \int_{-1}^{1} (1 - \theta_2^2 - \theta_3^2)^{1/2}\, \mathrm{d}\xi \tag{7}$$

The tangent unit vector of the deformed centroid axis at point P may be expressed as

$$\boldsymbol{t} = \{\cos\theta_n, \theta_3, -\theta_2\} \tag{8}$$

where

$$\cos\theta_n = (1 - \theta_2^2 - \theta_3^2)^{1/2} \tag{9}$$

and θ_2 and θ_3 are defined in Eqs. (5) and (6), respectively.

Let $\boldsymbol{e}_i$ and $\boldsymbol{e}_i^s (i = 1, 2, 3)$ denote the unit vectors associated with the x_i and x_i^s axes, respectively. Note that x_i axes coincide with x_i^s axes in the undeformed beam, and $\boldsymbol{e}_1^s$ coincides with the vector $\boldsymbol{t}$ in Eq. (8). In this study, the triad $\boldsymbol{e}_i^s$ in the deformed state is assumed to be achieved by the successive application of the following two rotation vectors to the triad $\boldsymbol{e}_i$:

$$\boldsymbol{\theta}_n = \theta_n \boldsymbol{n} \tag{10}$$

and

$$\boldsymbol{\theta}_t = \theta_1 \boldsymbol{t} \tag{11}$$

where $\boldsymbol{t}$ is defined in Eq. (8), θ_n is the inverse of $\cos\theta_n$ given in Eq. (9), and $\boldsymbol{n}$ is the unit vector perpendicular to the vectors $\boldsymbol{e}_1$ and $\boldsymbol{t}$ and given by

$$\boldsymbol{n} = \{0, \theta_2/(\theta_2^2 + \theta_3^2)^{1/2}, \theta_3/(\theta_2^2 + \theta_3^2)^{1/2}\}$$

$$= \{0, n_2, n_3\} \tag{12}$$

As can be seen, $\boldsymbol{\theta}_n$ in Eq. (10) and $\boldsymbol{\theta}_t$ in Eq. (11) are determined by $\theta_i (i = 1, 2, 3)$. Thus, θ_i are called rotation parameters in this study. The triad $\boldsymbol{e}_i$ are rotated about the axis with unit vector $\boldsymbol{n}$ by the application of the rotation vector $\boldsymbol{\theta}_n$ to an intermediate position $\boldsymbol{e}_i'$. Note that the unit vector $\boldsymbol{e}_1'$ coincides with the unit vector $\boldsymbol{t}$. Then the triad $\boldsymbol{e}_i'$ are rotated about the axis with unit vector $\boldsymbol{t}$ by the application of the rotation vector $\boldsymbol{\theta}_t$ to their final positions $\boldsymbol{e}_i^s$. It should be noted that the triad $\boldsymbol{e}_i^s$ are uniquely determined for a given set of triad $\boldsymbol{e}_i$ and rotation vectors $\boldsymbol{\theta}_n$ and $\boldsymbol{\theta}_t$; the rotation vectors $\boldsymbol{\theta}_n$ and $\boldsymbol{\theta}_t$ are uniquely determined for a given set of $\boldsymbol{e}_i$ and $\boldsymbol{e}_i^s$.

By using Eqs. (1), (10), (11), and (12) to the vectors $\boldsymbol{e}_i$ ($i = 1, 2, 3$), the relation between the vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_i^s$ in the element coordinate system may be obtained as

$$\boldsymbol{e}_i^s = [\boldsymbol{t}, \boldsymbol{R}_1, \boldsymbol{R}_2]\boldsymbol{e}_i = \boldsymbol{R}\boldsymbol{e}_i$$

$$\boldsymbol{R}_1 = \cos\theta_1 \boldsymbol{R}_a + \sin\theta_1 \boldsymbol{R}_b$$

$$\boldsymbol{R}_2 = -\sin\theta_1 \boldsymbol{R}_a + \cos\theta_1 \boldsymbol{R}_b$$

$$\boldsymbol{R}_a = \{-\theta_3, \cos\theta_n + (1 - \cos\theta_n)n_2^2, (1 - \cos\theta_n)n_2 n_3\}$$

$$\boldsymbol{R}_b = \{\theta_2, (1 - \cos\theta_n)n_2 n_3, \cos\theta_n + (1 - \cos\theta_n)n_3^2\} \tag{13}$$

where $\boldsymbol{R}$ is the so-called rotation matrix.

Because the rotation matrix $\boldsymbol{R}$ is a function of rotation parameters θ_i ($i = 1, 2, 3$), the variation of Eq. (13) may be written as

$$\delta e_i^s = e_i^s(\boldsymbol{\theta} + \delta\boldsymbol{\theta}) - e_i^s(\boldsymbol{\theta}) = [\boldsymbol{R}(\boldsymbol{\theta} + \delta\boldsymbol{\theta}) - \boldsymbol{R}(\boldsymbol{\theta})]\boldsymbol{e}_i$$
$$= \delta\boldsymbol{R}\boldsymbol{e}_i = \delta\boldsymbol{R}\boldsymbol{R}^t\boldsymbol{e}_i^s \tag{14}$$

where $\boldsymbol{\theta} = \{\theta_1, \theta_2, \theta_3\}$ is the vector of rotation parameters; $\delta\boldsymbol{\theta}$ and $\delta\boldsymbol{R}$ are the variation of $\boldsymbol{\theta}$ and $\boldsymbol{R}$, respectively. Since $\boldsymbol{R}\boldsymbol{R}^t = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix of order 3 × 3, $\delta\boldsymbol{R}\boldsymbol{R}^t$ is an antisymmetric matrix. There exists, therefore, a vector $\delta\boldsymbol{\phi} = \{\delta\phi_1, \delta\phi_2, \delta\phi_3\}$ satisfying $\delta\boldsymbol{R}\boldsymbol{R}^t = \delta\boldsymbol{\phi} \times \boldsymbol{I}$,[10] where $\delta\phi_i$ are infinitesimal rotations about x_i axes. Through some straightforward algebra, the relationship between $\delta\boldsymbol{\phi}$ and $\delta\boldsymbol{\theta}$ may be expressed as

$$\delta\boldsymbol{\phi} = [\boldsymbol{t}, (\boldsymbol{r}_1 + \alpha\boldsymbol{t}), (\boldsymbol{r}_2 + \beta\boldsymbol{t})]\delta\boldsymbol{\theta} = \boldsymbol{T}\delta\boldsymbol{\theta} \tag{15}$$

where

$$\boldsymbol{r}_1 = \left[-\theta_3, \frac{1-\theta_3^2}{\cos\theta_n}, \frac{\theta_2\theta_3}{\cos\theta_n}\right], \quad \boldsymbol{r}_2 = \left[\theta_2, \frac{\theta_2\theta_3}{\cos\theta_n}, \frac{1-\theta_2^2}{\cos\theta_n}\right]$$

$$\alpha = \frac{\theta_3}{\theta_2^2 + \theta_3^2}(1 - \cos\theta_n), \quad \beta = \frac{-\theta_2}{\theta_2^2 + \theta_3^2}(1 - \cos\theta_n) \tag{16}$$

The inverse relationship to that of Eq. (15) can be obtained algebraically and expressed as

$$\delta\boldsymbol{\theta} = \begin{bmatrix} 1 & \alpha & \beta \\ -\theta_3 & \cos\theta_n & 0 \\ \theta_2 & 0 & \cos\theta_n \end{bmatrix}\delta\boldsymbol{\phi} = \boldsymbol{T}^{-1}\delta\boldsymbol{\phi} \tag{17}$$

Note that if the rotational parameters θ_2 and θ_3 are much smaller than unity, α and β may be approximated by $^1/_2\theta_3$ and $-^1/_2\theta_2$, respectively.

Nodal Parameters and Nodal Forces

The global nodal parameters for the system of equations associated with the individual elements are chosen to be ΔU_{ij}, the X_i ($i = 1, 2, 3$) components of the incremental translation vectors $\Delta\boldsymbol{U}_j$ at nodes j ($j = 1, 2$), and $\Delta\Phi_{ij}$, the X_i ($i = 1, 2, 3$) components of the incremental rotation vectors $\Delta\boldsymbol{\Phi}_j$ at nodes j ($j = 1, 2$). When $\Delta\Phi_{ij}$ approach zero, $\Delta\Phi_{ij}$ represent infinitesimal rotations about the X_i axes. Thus, the generalized nodal forces corresponding to $\Delta\Phi_{ij}$ are the conventional moments about the X_i axes. The nodal forces corresponding to ΔU_{ij} are the forces in the X_i directions.

The beam element developed here has two nodes, which are located at shear centers of the two end cross sections of the beam element, with six degrees of freedom per node. Two sets of element nodal parameters termed "explicit nodal parameters" and "implicit nodal parameters" are employed. The explicit nodal parameters of the element are used in the assembly of the system matrices from the element matrices; thus they must be consistent with the global nodal parameters and are chosen to be Δu_{ij}, the x_i ($i = 1, 2, 3$) components of the incremental translation vectors $\Delta\boldsymbol{u}_j$ at nodes j ($j = 1, 2$), and $\Delta\phi_{ij}$, the x_i ($i = 1, 2, 3$) components of the incremental rotation vectors $\Delta\boldsymbol{\phi}_j$ at nodes j ($j = 1, 2$). The incremental rotational vectors $\Delta\boldsymbol{\phi}_j$ are applied to the axes of the element end cross section coordinates to update their orientations. Following the arguments concerning the generalized nodal forces corresponding to the global nodal parameters, the generalized nodal forces corresponding to element explicit nodal parameters $\Delta\phi_{ij}$ and Δu_{ij} are m_{ij} and f_{ij}, the conventional moments about the x_i axes, and the forces in the x_i directions, respectively.

The implicit nodal parameters of the element, which are used to determine the deformation of the beam element, are chosen to be u_{ij}, the x_i ($i = 1, 2, 3$) components of the total displacement vectors $\boldsymbol{u}_j$ at nodes j ($j = 1, 2$), and θ_{ij}, the nodal values of rotation parameters θ_i ($i = 1, 2, 3$) at nodes j ($j = 1, 2$), where θ_2 and θ_3 are defined in Eqs. (5) and (6) and θ_1 is defined in Eq. (11). The generalized nodal forces corresponding to the variations of u_{ij} are f_{ij}, the forces in the x_i directions, and those corresponding to the variations of θ_{ij} are generalized moments m_{ij}^θ, which are not conventional moments, because the variations of θ_{ij} are not infinitesimal rotations about the fixed axes x_i.

In view of Eq. (17), the relations between the variation of the implicit and explicit nodal parameters may be expressed as

$$\delta\boldsymbol{q}_\theta = \begin{Bmatrix} \delta\boldsymbol{u}_1 \\ \delta\boldsymbol{\theta}_1 \\ \delta\boldsymbol{u}_2 \\ \delta\boldsymbol{\theta}_2 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}_1^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{T}_2^{-1} \end{bmatrix} \begin{Bmatrix} \delta\boldsymbol{u}_1 \\ \delta\boldsymbol{\phi}_1 \\ \delta\boldsymbol{u}_2 \\ \delta\boldsymbol{\phi}_2 \end{Bmatrix} = \boldsymbol{T}_{\theta q}\delta\boldsymbol{q} \tag{18}$$

where $\boldsymbol{T}_j^{-1}$ are nodal values of $\boldsymbol{T}^{-1}$, $\delta\boldsymbol{u}_j = \{\delta u_{1j}, \delta u_{2j}, \delta u_{3j}\}$, $\delta\boldsymbol{\theta}_j = \{\delta\theta_{1j}, \delta\theta_{2j}, \delta\theta_{3j}\}$, and $\delta\boldsymbol{\phi}_j = \{\delta\phi_{1j}, \delta\phi_{2j}, \delta\phi_{3j}\}$, $j = 1, 2$. The identity and zero matrices of order 3 × 3 are $\boldsymbol{I}$ and $\boldsymbol{0}$, respectively.

Let $\boldsymbol{f} = \{\boldsymbol{f}_1, \boldsymbol{m}_1, \boldsymbol{f}_2, \boldsymbol{m}_2\}$, $\boldsymbol{f}_\theta = \{\boldsymbol{f}_1, \boldsymbol{m}_1^\theta, \boldsymbol{f}_2, \boldsymbol{m}_2^\theta\}$, where $\boldsymbol{f}_j = \{f_{1j}, f_{2j}, f_{3j}\}$, $\boldsymbol{m}_j = \{m_{1j}, m_{2j}, m_{3j}\}$, and $\boldsymbol{m}_j^\theta = \{m_{1j}^\theta, m_{2j}^\theta, m_{3j}^\theta\}$, $j = 1, 2$, denote the vectors of internal nodal forces corresponding to the variation of the explicit and implicit nodal parameters, $\delta\boldsymbol{q}$ and $\delta\boldsymbol{q}_\theta$, respectively. Using the contragradient law[24] and Eq. (18), the relation between $\boldsymbol{f}$ and $\boldsymbol{f}_\theta$ may be given by

$$\boldsymbol{f} = \boldsymbol{T}_{\theta q}^t\boldsymbol{f}_\theta \tag{19}$$

Determination of Element End Cross Section Coordinates, Element Coordinates, and Element Implicit Nodal Parameters

Assume that the incremental iterative method is used for the solution of nonlinear equilibrium equations and the configuration I is known. Here the configuration I may denote the equilibrium configuration of the previous increment or the configuration of the previous iteration. Let ${}^I\boldsymbol{X}_j$ and ${}^Ix_{ij}^s$ ($j = 1, 2$) denote the node coordinate vectors and element end cross section coordinates of an element at node j corresponding to configuration I. Let $\Delta\boldsymbol{U}_j$ and $\Delta\boldsymbol{\Phi}_j$ ($j = 1, 2$) denote increments of displacement and rotation vectors of an element at nodes j extracted from the global nodal parameter increments of the system of equations. In this paper, when the configuration I denotes the equilibrium configuration of the previous increment, $\Delta\boldsymbol{U}_j$ and $\Delta\boldsymbol{\Phi}_j$ denote the nodal displacement and rotation increments between two successive increments; when the configuration I denotes the configuration of the previous iteration, $\Delta\boldsymbol{U}_j$ and $\Delta\boldsymbol{\Phi}_j$ denote the nodal displacement and rotation increments between two successive iterations.

The way to determine the current element end cross section coordinates, element coordinates, and element internal nodal parameters corresponding to $\Delta\boldsymbol{U}_j$ and $\Delta\boldsymbol{\Phi}_j$ ($j = 1, 2$) is given as follows:

The current node coordinate vectors can be obtained by adding $\Delta\boldsymbol{U}_j$ to ${}^I\boldsymbol{X}_j$. Then the current x_1 axis of the element coordinate system can be determined by the line passing through nodes 1 and 2. The current element end cross section coordinates x_{ij}^s are obtained by the application of the rotation vectors $\Delta\boldsymbol{\Phi}_j$ ($j = 1, 2$) to the coordinate axes ${}^Ix_{ij}^s$ ($i = 1, 2, 3$). Let $\boldsymbol{\theta}_{nj}$ denote the nodal values of the rotation vector $\boldsymbol{\theta}_n$ defined in Eq. (10) at nodes j ($j = 1, 2$), which may be given by

$$\boldsymbol{\theta}_{nj} = \cos^{-1}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_{1j}^s)\frac{\boldsymbol{e}_1 \times \boldsymbol{e}_{1j}^s}{\|\boldsymbol{e}_1 \times \boldsymbol{e}_{1j}^s\|} \tag{20}$$

where $\boldsymbol{e}_{1j}^s$ and $\boldsymbol{e}_1$ are unit vectors associated with the x_{1j}^s and x_1 axes, respectively. The x_2 and x_3 axes of the element coordinate system are determined by the following two steps.

Step 1: The rotation vectors $-\boldsymbol{\theta}_{nj}$ are applied to the x_{ij}^{s} axes (Fig. 3a), respectively. The resultant coordinate axes are labeled $x_{ij}^{s'}$ axes (Fig. 3b). As can be seen, the $x_{1j}^{s'}$ axes coincide with the x_1 axis and the $x_{2j}^{s'}$ and $x_{3j}^{s'}$ axes are perpendicular to the x_1 axis.

Step 2: The unit vectors $\boldsymbol{e}_i$ (Fig. 3b) associated with x_i (i = 2, 3) axes of the element coordinates are defined as

$$\boldsymbol{e}_i = \frac{\boldsymbol{e}_{i1}^{s'} + \boldsymbol{e}_{i2}^{s'}}{\|\boldsymbol{e}_{i1}^{s'} + \boldsymbol{e}_{i2}^{s'}\|} \tag{21}$$

where $\boldsymbol{e}_{ij}^{s'}$ (i = 2, 3, j = 1, 2) are the unit vectors in the directions of $x_{ij}^{s'}$ axes.

The implicit nodal parameters θ_{2j} and θ_{3j} corresponding to $\boldsymbol{\theta}_{nj}$ given in Eq. (20) can be determined from Eqs. (9), (10), and (12). However, based on the assumption of small deformation, the x_i (i = 2, 3) components of the rotation vectors $\boldsymbol{\theta}_{nj}$ may be used as θ_{2j} and θ_{3j}. Let $\theta_{1j}\boldsymbol{e}_1$ (j = 1, 2) denote the rotation vectors that could rotate the x_2 axis to the $x_{2j}^{s'}$ axes, respectively. From Eq. (11) and the definition of implicit nodal parameters, it is clear that θ_{1j} are the implicit nodal parameters to be determined. Due to the definition of the element coordinate system, except u_{12}, all of the implicit nodal variables u_{ij} (i = 1, 2, 3, j = 1, 2) are identical to zero. The values of u_{12} can be obtained from the difference between the current and the original chord length of the beam element.

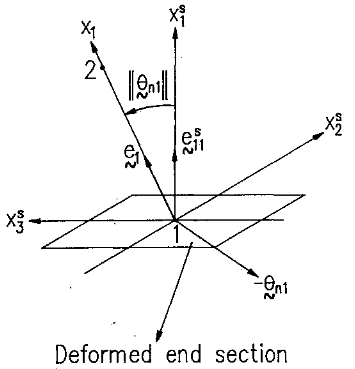

Fig. 3a Step 1 for the determination of x_2 and x_3 axes of the element coordinates.

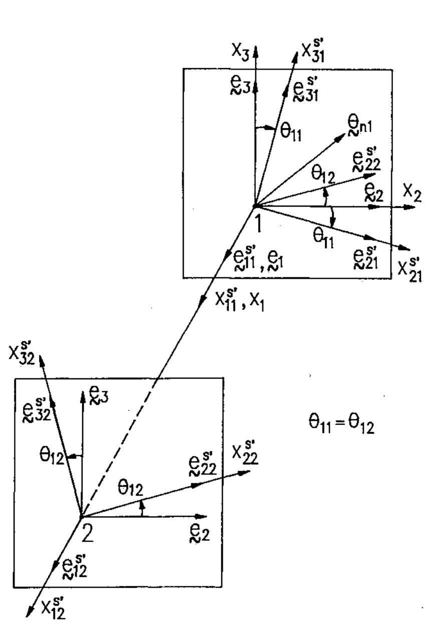

Fig. 3b Step 2 for the determination of x_2 and x_3 axes of the element coordinates.

Kinematics of Beam Elements

The deformations of the beam element are described in the element coordinate system. From the assumptions made at the beginning of this section, the deformations of the beam element may be determined by the displacements of the centroid axis of the beam element, orientation of cross section (element cross section coordinate system), and the out-of-plane warping of the cross section. The position vector $\boldsymbol{r}$ of an arbitrary point Q (Fig. 1) in the deformed beam, which has position vector $\boldsymbol{r}_0 = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$ in the undeformed state, may be written as

$$\boldsymbol{r} = x_c(s)\boldsymbol{e}_1 + v(s)\boldsymbol{e}_2 + w(s)\boldsymbol{e}_3 + y\boldsymbol{e}_2^{s} + z\boldsymbol{e}_3^{s} + \theta_{1,s}\omega\boldsymbol{e}_1^{s} \tag{22}$$

where $x_c(s)$, $v(s)$, and $w(s)$ are the x_i (i = 1, 2, 3) coordinates, respectively, of the point (x, 0, 0) on the undeformed centroid axis as mentioned; $\theta_{1,s}$ is the twist rate of the cross section along the deformed centroid axis, θ_1 is defined in Eq. (11), and $\omega = \omega(y,z)$ is the Saint Venant warping function for a prismatic beam of the same cross section.

Here, $v(s)$ and $w(s)$ in Eq. (22) are assumed to be the cubic Hermitian polynomials, and θ_1 in Eq. (11) is assumed to be linear polynomials along the deformed centroid axis of the beam element and may be given by

$$v(s) = \{N_1, N_2, N_3, N_4\}^t\{u_{21}, \theta_{31}, u_{22}, \theta_{32}\} = \boldsymbol{N}_b^t\boldsymbol{u}_b \tag{23}$$

$$w(s) = \{N_1, -N_2, N_3, -N_4\}^t\{u_{31}, \theta_{21}, u_{32}, \theta_{22}\} = \boldsymbol{N}_c^t\boldsymbol{u}_c \tag{24}$$

$$\theta_1(s) = \{N_5, N_6\}^t\{\theta_{11}, \theta_{12}\} = \boldsymbol{N}_d^t\boldsymbol{u}_d \tag{25}$$

where u_{2j} and u_{3j} (j = 1, 2) are the nodal values of v and w at nodes j, respectively; θ_{ij} are the nodal values of θ_i (i = 1, 2, 3) at nodes j (j = 1, 2), respectively; N_i (i = 1–6) are shape functions and are given by

$$N_1 = {}^1\!/\!_4(1-\xi)^2(2+\xi), \quad N_2 = \frac{S}{8}(1-\xi^2)(1-\xi)$$

$$N_3 = {}^1\!/\!_4(1+\xi)^2(2-\xi), \quad N_4 = \frac{S}{8}(-1+\xi^2)(1+\xi)$$

$$N_5 = {}^1\!/\!_2(1-\xi), \qquad N_6 = {}^1\!/\!_2(1+\xi) \tag{26}$$

where S is the current arc length of the centroid axis, and $\xi = -1 + 2(s/S)$.

The displacement vector of point Q is represented as

$$\boldsymbol{u} = \boldsymbol{r} - \boldsymbol{r}_0 \tag{27}$$

Substitution of Eqs. (4), (7), (13), and (22) into Eq. (27), and use of the approximation $\cos\theta_n = 1 - {}^1\!/\!_2\theta_2^2 - {}^1\!/\!_2\theta_3^2$, the x_i (i = 1, 2, 3) components of the displacement vector $\boldsymbol{u} = \{u_1, u_2, u_3\}$ may be expressed as

$$u_1 = u - y(\theta_3\cos\theta_1 - \theta_2\sin\theta_1) + z(\theta_2\cos\theta_1 + \theta_3\sin\theta_1) + \theta_{1,s}\omega \tag{28}$$

$$u_2 = v + y[(1 - {}^1\!/\!_2\theta_3^2)\cos\theta_1 + {}^1\!/\!_2\theta_2\theta_3\sin\theta_1 - 1] + z[{}^1\!/\!_2\theta_2\theta_3\cos\theta_1 - (1 - {}^1\!/\!_2\theta_3^2)\sin\theta_1] + \theta_{1,s}\theta_3\omega \tag{29}$$

$$u_3 = w + y[{}^1\!/\!_2\theta_2\theta_3\cos\theta_1 + (1 - {}^1\!/\!_2\theta_2^2)\sin\theta_1] + z[(1 - {}^1\!/\!_2\theta_2^2)\cos\theta_1 - {}^1\!/\!_2\theta_2\theta_3\sin\theta_1 - 1] - \theta_{1,s}\theta_2\omega \tag{30}$$

where

$$u = u_{11} + \ell\left\{\frac{1+\xi}{2}\left[1 + \frac{1}{4}\int_{-1}^{1}(\theta_2^2 + \theta_3^2)\,\mathrm{d}\xi\right] - \frac{1}{4}\int_{-1}^{\xi}(\theta_2^2 + \theta_3^2)\,\mathrm{d}\xi\right\} - x \tag{31}$$

In this study, the Green strains are used for the measure of strain. Using assumption 4, we only consider the strain components ϵ_{11}, ϵ_{12}, and ϵ_{13}, which are given by

$$\epsilon_{11} = u_{1,x} + {}^1\!/_2(u_{1,x}^2 + u_{2,x}^2 + u_{3,x}^2)$$

$$\epsilon_{12} = {}^1\!/_2(u_{1,y} + u_{2,x}) + {}^1\!/_2(u_{1,x}u_{1,y} + u_{2,x}u_{2,y} + u_{3,x}u_{3,y})$$

$$\epsilon_{13} = {}^1\!/_2(u_{1,z} + u_{3,x}) + {}^1\!/_2(u_{1,x}u_{1,z} + u_{2,x}u_{2,z} + u_{3,x}u_{3,z}) \qquad (32)$$

Using the chain rule for differentiation and Eq. (2), $(\ )_{,x}$ in Eq. (32) may be expressed as

$$(\ )_{,x} = (\ )_{,s}(1 + \epsilon_0) \qquad (33)$$

Substituting Eqs. (28–31) and (33) into Eq. (32) and neglecting terms of third and higher orders (i.e., products of three displacements or their derivatives) yield

$$\epsilon_{11} = \epsilon_0 + {}^1\!/_2\epsilon_0^2 + (1 + \epsilon_0)^2[y(-\theta_{3,s}\cos\theta_1 + \theta_{2,s}\sin\theta_1)$$
$$+ z(\theta_{2,s}\cos\theta_1 + \theta_{3,s}\sin\theta_1) + {}^1\!/_2y^2(\theta_{1,s}^2 + \theta_{3,s}^2)\cos^2\theta_1$$
$$+ {}^1\!/_2z^2(\theta_{1,s}^2 + \theta_{2,s}^2)\cos^2\theta_1 - yz\theta_{2,s}\theta_{3,s}\cos^2\theta_1] \qquad (34)$$

$$\epsilon_{12} = {}^1\!/_2(1 + \epsilon_0)[(-z + \omega_{,y})\theta_{1,s} + {}^1\!/_2z(\theta_2\theta_{3,s} - \theta_3\theta_{2,s})$$
$$+ \omega_{,y}\theta_{1,s}(z\theta_{2,s} - y\theta_{3,s}) + \theta_{1,s}\theta_{3,s}\omega] \qquad (35)$$

$$\epsilon_{13} = {}^1\!/_2(1 + \epsilon_0)[(y + \omega_{,z})\theta_{1,s} + {}^1\!/_2y(\theta_3\theta_{2,s} - \theta_2\theta_{3,s})$$
$$+ \omega_{,z}\theta_{1,s}(z\theta_{2,s} - y\theta_{3,s}) - \theta_{1,s}\theta_{2,s}\omega] \qquad (36)$$

with

$$\epsilon_0 = \frac{\ell}{L}\left[1 + {}^1\!/_4\int_{-1}^{1}(\theta_2^2 + \theta_3^2)\,d\xi\right] - 1 \qquad (37)$$

Element Nodal Force Vectors

The element nodal force vectors corresponding to the implicit nodal parameters are obtained from the virtual work principle. For convenience, the implicit nodal parameters are divided into four generalized nodal displacement vectors, $\boldsymbol{u}_i$ ($i = a, b, c, d$), where

$$\boldsymbol{u}_a = \{u_{11}, u_{12}\} \qquad (38)$$

and $\boldsymbol{u}_b$, $\boldsymbol{u}_c$, and $\boldsymbol{u}_d$ are defined in Eqs. (23), (24), and (25), respectively. The generalized force vectors corresponding to $\delta\boldsymbol{u}_i$, the variation of $\boldsymbol{u}_i$ ($i = a, b, c, d$) are

$$\boldsymbol{f}_a = \{f_{11}, f_{12}\}, \quad \boldsymbol{f}_b = \{f_{21}, m_{31}^\theta, f_{22}, m_{32}^\theta\}$$

$$\boldsymbol{f}_c = \{f_{31}, m_{21}^\theta, f_{32}, m_{22}^\theta\}, \quad \boldsymbol{f}_d = \{m_{11}^\theta, m_{12}^\theta\} \qquad (39)$$

where f_{ij} are the forces in the x_i directions and m_{ij}^θ are the generalized moments as mentioned before.

The principle of virtual work requires that

$$\delta\boldsymbol{u}_a^t\boldsymbol{f}_a + \delta\boldsymbol{u}_b^t\boldsymbol{f}_b + \delta\boldsymbol{u}_c^t\boldsymbol{f}_c + \delta\boldsymbol{u}_d^t\boldsymbol{f}_d$$
$$= \int_v (\sigma_{11}\delta\epsilon_{11} + 2\sigma_{12}\delta\epsilon_{12} + 2\sigma_{13}\delta\epsilon_{13})\,dv \qquad (40)$$

where v is the volume of the undeformed beam element, $\delta\epsilon_{1j}$ ($j = 1, 2, 3$) are the variation of ϵ_{1j} in Eqs. (34–36), respectively, with respect to the implicit nodal parameters. The second Piola-Kirchhoff stresses are σ_{11}, σ_{12}, and σ_{13}. For linear elastic material, the following constitutive equations are used:

$$\sigma_{11} = E\epsilon_{11}, \quad \sigma_{12} = 2G\epsilon_{12}, \quad \sigma_{13} = 2G\epsilon_{13} \qquad (41)$$

in which E is the Young's modulus, and G is the shear modulus.

Substituting Eqs. (23–25), (34–36), and (41) into Eq. (40), retaining the terms up to the second order of nodal parameters, and equating the coefficients of $\delta\boldsymbol{u}_i$ ($i = a, b, c, d$) on both sides of Eq. (40), we may obtain

$$\boldsymbol{f}_a = \boldsymbol{G}_a\,[AEL(\epsilon_0 + {}^3\!/_2\epsilon_0^2) + EI_z\textstyle\int({}^5\!/_2\theta_{3,s}^2 - \theta_{3,s}\theta_{3,s}^*)\,ds$$
$$+ EI_y\textstyle\int({}^5\!/_2\theta_{2,s}^2 - \theta_{2,s}\theta_{2,s}^*)\,ds + {}^1\!/_2EI_p\textstyle\int\theta_{1,s}^2\,ds] \qquad (42)$$

$$\boldsymbol{f}_b = AEL(1 + \epsilon_0)\epsilon_0\boldsymbol{G}_b + EI_z(1 + 4\epsilon_0)\textstyle\int\boldsymbol{N}_b''\theta_{3,s}\,ds$$
$$- E(I_z - I_y)\textstyle\int\boldsymbol{N}_b''\theta_1\theta_{2,s}\,ds - E\alpha_y\textstyle\int\boldsymbol{N}_b''({}^1\!/_2\theta_{1,s}^2$$
$$+ {}^3\!/_2\theta_{3,s}^2)\,ds + 3E\alpha_{yz}\textstyle\int\boldsymbol{N}_b''\theta_{2,s}\theta_{3,s}\,ds$$
$$- E\alpha_{zy}\textstyle\int\boldsymbol{N}_b''({}^1\!/_2\theta_{1,s}^2 + {}^3\!/_2\theta_{2,s}^2)\,ds - \frac{GJ}{2}\textstyle\int\boldsymbol{N}_b''\theta_{1,s}\theta_2\,ds$$
$$+ \frac{GJ}{2}\textstyle\int\boldsymbol{N}_b'\theta_{1,s}\theta_{2,s}\,ds + 2GJ_2\textstyle\int\boldsymbol{N}_b''\theta_{1,s}^2\,ds \qquad (43)$$

$$\boldsymbol{f}_c = AEL(1 + \epsilon_0)\epsilon_0\boldsymbol{G}_c - EI_y(1 + 4\epsilon_0)\textstyle\int\boldsymbol{N}_c''\theta_{2,s}\,ds$$
$$+ E(I_z - I_y)\textstyle\int\boldsymbol{N}_c''\theta_1\theta_{3,s}\,ds - E\alpha_{yz}\textstyle\int\boldsymbol{N}_c''({}^1\!/_2\theta_{1,s}^2$$
$$+ {}^3\!/_2\theta_{3,s}^2)\,ds + 3E\alpha_{zy}\textstyle\int\boldsymbol{N}_c''\theta_{2,s}\theta_{3,s}\,ds$$
$$- E\alpha_z\textstyle\int\boldsymbol{N}_c''({}^1\!/_2\theta_{1,s}^2 + {}^3\!/_2\theta_{2,s}^2)\,ds - \frac{GJ}{2}\textstyle\int\boldsymbol{N}_c''\theta_{1,s}\theta_3\,ds$$
$$+ \frac{GJ}{2}\textstyle\int\boldsymbol{N}_c'\theta_{1,s}\theta_{3,s}\,ds + 2GJ_1\textstyle\int\boldsymbol{N}_c''\theta_{1,s}^2\,ds \qquad (44)$$

$$\boldsymbol{f}_d = GJ(1 + \epsilon_0)\textstyle\int\boldsymbol{N}_d'\theta_{1,s}\,ds - \frac{GJ}{2}\textstyle\int\boldsymbol{N}_d'(\theta_2\theta_{3,s} - \theta_3\theta_{2,s})\,ds$$
$$- 4GJ_1\textstyle\int\boldsymbol{N}_d'\theta_{1,s}\theta_{2,s}\,ds + 4GJ_2\textstyle\int\boldsymbol{N}_d'\theta_{1,s}\theta_{3,s}\,ds$$
$$- E(I_z - I_y)\textstyle\int\boldsymbol{N}_d\theta_{2,s}\theta_{3,s}\,ds + EI_p\epsilon_0\textstyle\int\boldsymbol{N}_d'\theta_{1,s}\,ds$$
$$- E(\alpha_y + \alpha_{zy})\textstyle\int\boldsymbol{N}_d'\theta_{1,s}\theta_{3,s}\,ds$$
$$+ E(\alpha_z + \alpha_{yz})\textstyle\int\boldsymbol{N}_d'\theta_{1,s}\theta_{2,s}\,ds \qquad (45)$$

where A is the cross section area, S is the current arc length of the centroid axis of the beam, the range of the integration for the integral $\int(\ )\,ds$ in Eqs. (42–45) is from 0 to S,

$$\boldsymbol{G}_a = \frac{1}{L}\{-1, 1\}, \quad \boldsymbol{G}_b = \frac{\ell}{2L}\int_{-1}^{1}\boldsymbol{N}_b'\theta_3\,d\xi,$$

$$\boldsymbol{G}_c = \frac{-\ell}{2L}\int_{-1}^{1}\boldsymbol{N}_c'\theta_2\,d\xi$$

$$\theta_{3,s}^* = \{2N_1'', N_2'', 2N_3'', N_4''\}^t\boldsymbol{u}_b = \boldsymbol{N}_b^{*'''}\boldsymbol{u}_b$$

$$\theta_{2,s}^* = \{2N_1'', -N_2'', 2N_3'', -N_4''\}^t\boldsymbol{u}_c = \boldsymbol{N}_c^{*'''}\boldsymbol{u}_c \qquad (46)$$

$$I_y = \textstyle\int z^2\,dA, \quad I_z = \int y^2\,dA, \quad I_p = I_y + I_z$$

$$\alpha_y = \textstyle\int y^3\,dA, \quad \alpha_{yz} = \int y^2z\,dA,$$

$$\alpha_{zy} = \textstyle\int yz^2\,dA, \quad \alpha_z = \int z^3\,dA$$

$$J = \textstyle\int[(-z + \omega_{,y})^2 + (y + \omega_{,z})^2]\,dA$$

$$J_1 = \textstyle\int(y + \omega_{,z})\omega\,dA, \quad J_2 = \int(-z + \omega_{,y})\omega\,dA \qquad (47)$$

It should be noted that because only the terms up to the second order of nodal parameters are retained in Eqs. (42–45) in this study, the terms of order greater than two in the expansion of $\boldsymbol{f}$ [Eq. (19)] are neglected.

Element Stiffness Matrix

The element tangent stiffness matrix corresponding to the explicit nodal parameters (referred to as explicit tangent stiffness matrix) may be obtained by differentiating the element nodal force vector $\boldsymbol{f}$ in Eq. (19) with respect to explicit nodal parameters. Using Eq. (19) in conjunction with Eqs. (17), (18), and (42–45), we obtain

$$\boldsymbol{k} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}_\theta}\frac{\partial \boldsymbol{q}_\theta}{\partial \boldsymbol{q}} = \boldsymbol{T}^t_{\theta q}\boldsymbol{k}_\theta \boldsymbol{T}_{\theta q} + \boldsymbol{H} \tag{48}$$

where $\boldsymbol{k}_\theta = (\partial \boldsymbol{f}_\theta / \partial \boldsymbol{q}_\theta)$ is the tangent stiffness matrix corresponding to the implicit nodal parameters (referred to as implicit tangent stiffness matrix), and $\boldsymbol{H}$ is a nonsymmetric matrix and is given by

$$\boldsymbol{H} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{H}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{H}_2 \end{bmatrix} \tag{49}$$

in which $\mathbf{0}$ is a zero matrix of order 3×3, and

$$\boldsymbol{H}_j = \begin{bmatrix} 0 & m^\theta_{3j} & -m^\theta_{2j} \\ 0 & 0 & {}^1\!/\!_2 m^\theta_{1j} \\ 0 & -{}^1\!/\!_2 m^\theta_{1j} & 0 \end{bmatrix} \tag{50}$$

Using the direct stiffness method,[24] the implicit tangent stiffness matrix $\boldsymbol{k}_\theta$ may be assembled by the submatrices

$$\boldsymbol{k}_{ij} = \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{u}_j} \tag{51}$$

where $\boldsymbol{f}_i$ $(i = a, b, c, d)$ are given in Eqs. (42–45), respectively, and $\boldsymbol{u}_j$ $(j = a, b, c, d)$ are defined in Eqs. (38) and (23–25), respectively. Note that because only the terms up to the second order of nodal parameters are retained in Eqs. (42–45), only the terms up to the first order in Eqs. (50) and (51) are retained. The explicit form of $\boldsymbol{k}_{ij}$ may be expressed as

$$\boldsymbol{k}_{aa} = AEL(1 + 3\epsilon_0)\boldsymbol{G}_a\boldsymbol{G}^t_a$$

$$\boldsymbol{k}_{ab} = \boldsymbol{k}^t_{ba} = AEL\boldsymbol{G}_a\boldsymbol{G}^t_b + EI_z\boldsymbol{G}_a[\int \boldsymbol{N}_b^{\prime\prime\prime t}(5\theta_{3,s} - \theta^*_{3,s})\,\mathrm{d}s - \int \boldsymbol{N}_b^{*\prime\prime\prime t}\theta_{3,s}\,\mathrm{d}s]$$

$$\boldsymbol{k}_{ac} = \boldsymbol{k}^t_{ca} = AEL\boldsymbol{G}_a\boldsymbol{G}^t_c - EI_y\boldsymbol{G}_a[\int \boldsymbol{N}_c^{\prime\prime\prime t}(5\theta_{2,s} - \theta^*_{2,s})\,\mathrm{d}s - \int \boldsymbol{N}_c^{*\prime\prime\prime t}\theta_{2,s}\,\mathrm{d}s]$$

$$\boldsymbol{k}_{ad} = \boldsymbol{k}^t_{da} = EI_p\boldsymbol{G}_a\int \boldsymbol{N}_d^{\prime\prime t}\theta_{1,s}\,\mathrm{d}s$$

$$\boldsymbol{k}_{bb} = \frac{AE\ell\epsilon_0}{2}\int_{-1}^{1} \boldsymbol{N}'_b\boldsymbol{N}_b^{\prime\prime t}\,\mathrm{d}\xi + EI_z(1 + 4\epsilon_0)\int \boldsymbol{N}''_b\boldsymbol{N}_b^{\prime\prime\prime t}\,\mathrm{d}s - 3E\alpha_y\int \boldsymbol{N}''_b\boldsymbol{N}_b^{\prime\prime\prime t}\theta_{3,s}\,\mathrm{d}s + 3E\alpha_{yz}\int \boldsymbol{N}''_b\boldsymbol{N}_b^{\prime\prime\prime t}\theta_{2,s}\,\mathrm{d}s$$

$$\boldsymbol{k}_{bc} = \boldsymbol{k}^t_{cb} = E(I_z - I_y)\int \boldsymbol{N}''_b\boldsymbol{N}_c^{\prime\prime\prime t}\theta_1\,\mathrm{d}s - 3E\alpha_{yz}\int \boldsymbol{N}''_b\boldsymbol{N}_c^{\prime\prime\prime t}\theta_{3,s}\,\mathrm{d}s + 3E\alpha_{zy}\int \boldsymbol{N}''_b\boldsymbol{N}_c^{\prime\prime\prime t}\theta_{2,s}\,\mathrm{d}s + \frac{GJ}{2}\int(\boldsymbol{N}''_b\boldsymbol{N}_c^{\prime\prime t} - \boldsymbol{N}'_b\boldsymbol{N}_c^{\prime\prime\prime t})\theta_{1,s}\,\mathrm{d}s$$

$$\boldsymbol{k}_{bd} = \boldsymbol{k}^t_{db} = -E(I_z - I_y)\int \boldsymbol{N}''_b\boldsymbol{N}^t_d\theta_{2,s}\,\mathrm{d}s - E(\alpha_y + \alpha_{zy})\int \boldsymbol{N}''_b\boldsymbol{N}_d^{\prime\prime t}\theta_{1,s}\,\mathrm{d}s - \frac{GJ}{2}\int \boldsymbol{N}''_b\boldsymbol{N}_d^{\prime\prime t}\theta_2\,\mathrm{d}s + \frac{GJ}{2}\int \boldsymbol{N}'_b\boldsymbol{N}_d^{\prime\prime t}\theta_{2,s}\,\mathrm{d}s + 4GJ_2\int \boldsymbol{N}''_b\boldsymbol{N}_d^{\prime\prime t}\theta_{1,s}\,\mathrm{d}s$$

$$\boldsymbol{k}_{cc} = \frac{AE\ell\epsilon_0}{2}\int_{-1}^{1} \boldsymbol{N}'_c\boldsymbol{N}_c^{\prime\prime t}d\xi + EI_y(1 + 4\epsilon_0)\int \boldsymbol{N}''_c\boldsymbol{N}_c^{\prime\prime\prime t}\,\mathrm{d}s - 3E\alpha_{zy}\int \boldsymbol{N}''_c\boldsymbol{N}_c^{\prime\prime\prime t}\theta_{3,s}\,\mathrm{d}s + 3E\alpha_z\int \boldsymbol{N}''_c\boldsymbol{N}_c^{\prime\prime\prime t}\theta_{2,s}\,\mathrm{d}s$$

$$\boldsymbol{k}_{cd} = \boldsymbol{k}^t_{dc} = E(I_z - I_y)\int \boldsymbol{N}''_c\boldsymbol{N}^t_d\theta_{3,s}\,\mathrm{d}s - E(\alpha_{yz} + \alpha_z)\int \boldsymbol{N}''_c\boldsymbol{N}_d^{\prime\prime t}\theta_{1,s}\,\mathrm{d}s - \frac{GJ}{2}\int \boldsymbol{N}''_c\boldsymbol{N}_d^{\prime\prime t}\theta_3\,\mathrm{d}s + \frac{GJ}{2}\int \boldsymbol{N}'_c\boldsymbol{N}_d^{\prime\prime t}\theta_{3,s}\,\mathrm{d}s + 4GJ_1\int \boldsymbol{N}''_c\boldsymbol{N}_d^{\prime\prime t}\theta_{1,s}\,\mathrm{d}s$$

$$\boldsymbol{k}_{dd} = GJ(1 + 2\epsilon_0)\int \boldsymbol{N}'_d\boldsymbol{N}_d^{\prime\prime t}\,\mathrm{d}s - 4GJ_1\int \boldsymbol{N}'_d\boldsymbol{N}_d^{\prime\prime t}\theta_{2,s}\,\mathrm{d}s + 4GJ_2\int \boldsymbol{N}'_d\boldsymbol{N}_d^{\prime\prime t}\theta_{3,s}\,\mathrm{d}s + EI_p\epsilon_0\int \boldsymbol{N}'_d\boldsymbol{N}_d^{\prime\prime t}\,\mathrm{d}s - E(\alpha_y + \alpha_{zy})\int \boldsymbol{N}'_d\boldsymbol{N}_d^{\prime\prime t}\theta_{3,s}\,\mathrm{d}s + E(\alpha_z + \alpha_{yz})\int \boldsymbol{N}'_d\boldsymbol{N}_d^{\prime\prime t}\theta_{2,s}\,\mathrm{d}s \tag{52}$$

III. Applications

An incremental-iterative method based on the Newton-Raphson method combined with constant arc length[8,21] is adopted for numerical studies. To improve the convergence properties of numerical iteration, the two-cycle iteration scheme introduced in Ref. 22 is also employed here. The adopted convergence criterion is

$$\rho = \frac{\|\varphi\|}{N\|\boldsymbol{P}\|} \leq \rho_{\mathrm{tol}} \tag{53}$$

where $\|\varphi\|$ is the Euclidean norm of the unbalanced forces, $\|\boldsymbol{P}\|$ is the Euclidean norm of the applied forces, N is the number of system of equations, and ρ_{tol} is a prescribed value of error tolerance.

The example considered is a cantilever beam subjected to an end load as shown in Fig. 4. Its geometry and material properties are[14,15,25]: $L = 0.508$ m, $b = 0.3175 \times 10^{-3}$ m, $h = 0.127 \times 10^{-1}$ m, $EI_z = 36.2695$ $N\text{-}m^2$, $EI_y = 2.2268$ $N\text{-}m^2$ (case A), $EI_y = 2.4783$ $N\text{-}m^2$ (case B), and $GJ = 2.9623$ $N\text{-}m^2$. The value of EI_y for case A is taken from Ref. 15, which was obtained by dividing EI_z by 16, based on the cross-sectional aspect ratio of 4. The value of EI_y for case B is taken from Ref. 14, which makes the first flatwise frequency with zero tip load match the experimental value given in Ref. 25. The present results are referred to as present-A and present-B for case A and case B, respectively.

The beam is idealized using 10 equal beam elements. The error tolerance is set to 10^{-4}. The present results are compared with the experimental results given in Ref. 25 and the numerical results given in Ref. 15.

The results for the end displacements V_{TIP}, W_{TIP}, and the tip twist angle $\bar{\phi}_{\mathrm{TIP}}$ (see the appendix of Ref. 15) vs applied load are shown in Figs. 5a–5c, respectively. It is seen that the present results, especially the results of present-B, are in excellent agreement with the experimental results. Figures 6a–6c show V_{TIP}, W_{TIP}, and $\bar{\phi}_{\mathrm{TIP}}$, respectively vs the loading angle for three values of the tip load. In all cases, very good agreement between the present results and the experimental results is obtained.

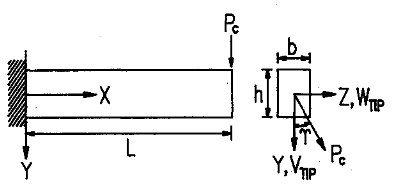

Fig. 4 Cantilever beam with an end load.

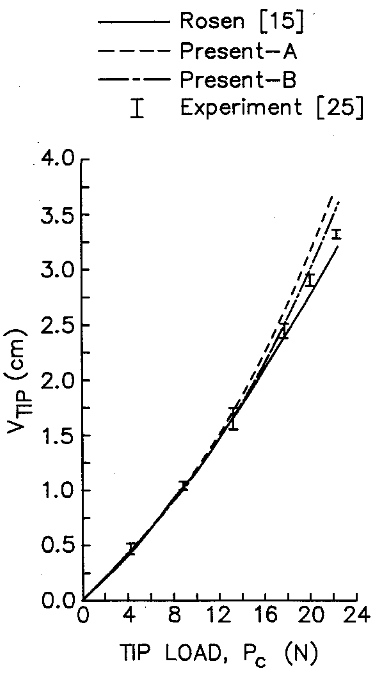

Fig. 5a Edgewise tip displacement vs tip load.

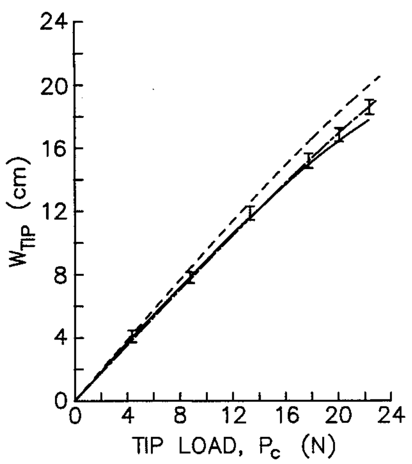

Fig. 5b Flatwise tip displacement vs tip load.

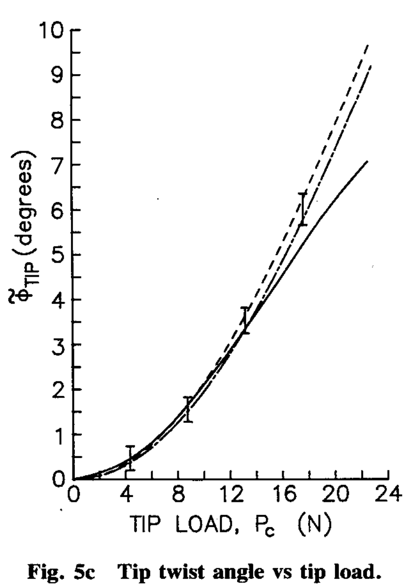

Fig. 5c Tip twist angle vs tip load.

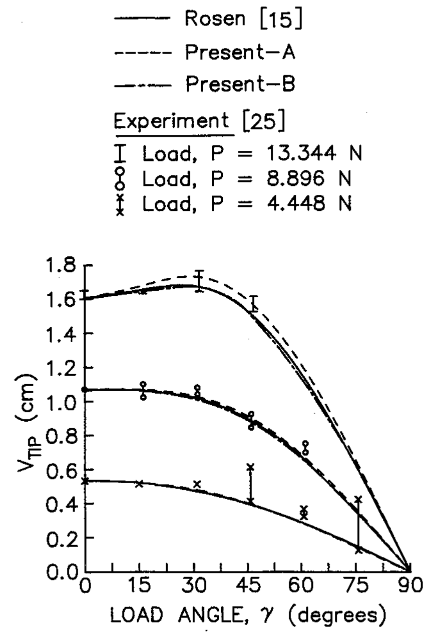

Fig. 6a Edgewise tip displacement vs loading angle.

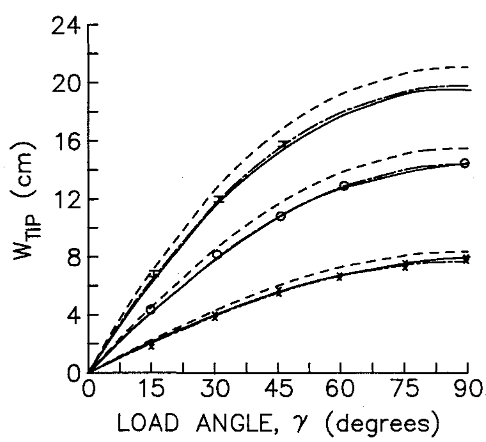

Fig. 6b Flatwise tip displacement vs loading angle.

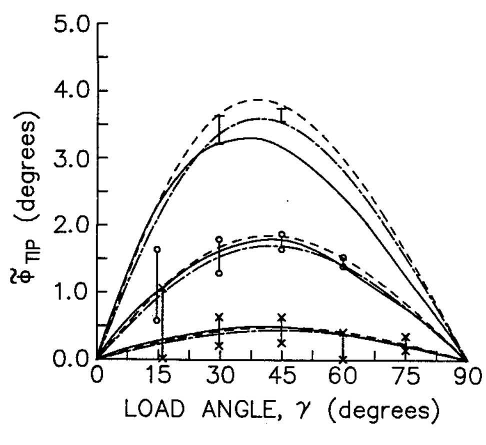

Fig. 6c Tip twist angle vs loading angle.

IV. Conclusions

A corotational total Lagrangian formulation of beam element is presented and applied to the nonlinear analysis of three-dimensional beam structures with large rotations but small strains. The nonlinear coupling among the bending, twisting, and stretching deformations is considered. A motion process and three rotation parameters are proposed to determine the orientation of the element cross section. The major geometric nonlinearities were shown to be embodied in the coordinate transformation when forming the element assemblage by the corotational formulation. The accuracy and efficiency of the proposed method are demonstrated by the results of numerical examples compared with the numerical and experimental results reported in the literature.

Despite the fact that the formulation of the beam element is relatively simple, highly accurate solutions are obtained. It is believed that the corotational total Lagrangian formulation of beam element proposed in this paper may represent a valuable engineering tool for the solution of nonlinear spatial beam structures.

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